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Stokes drift in two-dimensional wave flumes

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A complete second-order solution is presented for the two-dimensional wave motion forced by a generic planar wavemaker. The wavemaker is doubly articulated and includes both piston and hinged wavemakers of variable draught. It is shown that the first-order evanescent eigenseries cannot be neglected when computing the amplitude of the second-order free wave. A previously neglected, time-independent solution that is required to satisfy an inhomogeneous kinematic boundary condition on the wavemaker as well as an inhomogeneous Neumann boundary condition on the free surface is examined in detail for the first time. This time-independent solution is found to accurately estimate the mean return flow in a closed wave flume computed by the Eulerian method. This mean return current due to Stokes drift is usually estimated using the principle of kinematic conservation of mass flux. Even though the first-order eigenseries will converge for any geometry of a generic planar wavemaker, the second-order solutions obtained from Stokes perturbation expansions will not converge for all planar wavemaker geometries.

1. Introduction

Havelock (1929) used Fourier integrals to develop a theory for forced surface gravity waves in water of both infinite and finite depth. Both planar and circular waves were treated. Kennard (1949) also used Fourier integrals but included the initial-value problem. Biesel & Suquet (1953) obtained explicit linear solutions for the wave motions generated by both a piston and a hinged wavemaker and Hyun (1976) later extended that work to include hinged wavemakers of variable draught. The solution presented by Hyun (1976) was extended by Hudspeth & Chen (1981) to a wave flumes consisting of two constant-depth regions connected by a gradually sloping transition region. Wave heights predicted by these wavemaker theories have been verified experimentally for piston wavemakers by Ursell, Dean & Yu (1960), Galvin (1964), and Keating & Webber (1977); and for hinged wavemakers of variable draught by Galvin (1964), Patel & Ionnaou (1980), and Hudspeth, Leonard & Chen (1981). When relatively long waves of finite amplitudes are generated by a sinusoidally moving wavemaker, it has been observed (Goda & Kikuya 1964; Multer & Galvin 1967; Iwagaki & Sakai 1970) that the propagating wave is not of permanent form as predicted by the linear solution, but rather breaks down into a primary wave and one or more secondary waves. The advancement of nonlinear wavemaker theories in both time and frequency domains has been stimulated, in part, because this secondary wave phenomenon is not predicted by the linear wavemaker solutions.

Fontanet (1961) developed a complete second-order nonlinear theory in Lagrangian coordinates for the waves generated by a sinusoidally moving plane wavemaker. However, his solution is relatively complicated to use and results are presented only for piston wavemakers. Madsen (1971) obtained a more useful approximate solution using a Stokes expansion for a piston wavemaker. However, his second-order solution neglected the effects of the first-order evanescent eigenmodes, so that the results are valid only for long waves. Multer (1973) solved the piston wavemaker problem numerically. Daugaard (1972) included the effects of the first-order evanescent eigenmodes using a Stokes expansion to obtain a second-order solution for a piston wavemaker. However, his second-order solution also neglected the effects of the first-order evanescent eigenmodes on the free-surface boundary conditions near the wavemaker and, accordingly, represents only an approximate solution to the complete second-order problem.

Flick & Guza (1980) analysed, using a Stokes expansion, the motion of a wavemaker that is hinged either on or below the channel bottom. They evaluated the relationship between the second-harmonic (secondary) waves forced by the wavemaker and the Stokes waves by computing the coefficients for the propagating eigenmode numerically. Analytical expressions for the coefficients in their second-order solution were not presented. However, their solution, like that of Daugaard (1972), also neglected the interactions of the first-order evanescent eigenmodes at the free-surface boundary near the wavemaker because they reasoned that these evanescent eigenmodes do not contribute to the propagating waves. Furthermore, their solution as well as those of Madsen (1971) and Daugaard (1972) is not complete because they neglected the time-independent, second-order solutions which are required to satisfy exactly the boundary conditions at the wavemaker and at the still water level.

Massel (1981) attempted to extend the work of Flick & Guza (1980) by including a time-independent solution for the kinematic boundary condition at the wavemaker. Since he neglected the effects of the first-order evanescent eigenmodes in the second-order boundary conditions, his time-independent solution does not satisfy the second-order boundary-value problem. A complete second-order solution is still needed in order to describe the nonlinear fluid motion generated by a planar wavemaker.

The theoretical and experimental investigations of nonlinear wavemaker-generated sloshing waves (cf. Kit, Shemer & Miloh 1987 for a recent list of references) or parametrically excited cross-waves generated by a plane wavemaker at a subharmonic frequency (cf. Miles & Becker 1988 for a recent list of references) are focused on the motion of the free surface in the far field and do not require the near-field information associated with either the near-field evanescent eigenmodes or the time-independent potentials obtained here.

In § 2, a complete analytical solution that is correct to second order in the wave slope is presented for the fluid motion forced by a sinusoidally moving generic planar wavemaker. The generic planar wavemaker is doubly articulated and includes both piston and hinged wavemakers of variable draught. Three time-dependent and two time-independent potentials are required in order to satisfy all of the boundary conditions at second-order. In §3, the mean horizontal momentum per unit area is computed using both the Eulerian and Lagrangian velocities.

2. Nonlinear wavemaker theory

For convenience, all physical variables (denoted by superscript asterisks) will be made dimensionless as follows:

$$(x, z, h, d, b, \Delta) = k^*(x^*, z^*, h^*, d^*, b^*, \Delta^*), \quad (t, T) = (g^*k^*)^{\frac{1}{2}}(t^*, T^*),$$

$$(\xi, \chi) = \frac{(\xi^*, \chi^*)}{S^*}, \quad (u, w) = \frac{(u^*, w^*)}{[A^*(g^*k^*)^{\frac{1}{2}}]}, \quad \Phi = \frac{\Phi^*}{[A^*(g^*/k^*)^{\frac{1}{2}}]},$$

$$\eta = \frac{\eta^*}{A^*}, \quad P = \frac{P^*}{\rho^*A^*g^*}, \quad B = \frac{B^*}{A^*g^*},$$

in which A^* is the amplitude of the first-harmonic wave component; k^* $(= 2\pi/L^*)$ the propagating wavenumber; L^* the wavelength; g^* the gravitational constant; ρ^* the fluid mass density; S^* the amplitude of the wavemaker stroke; and T^* the wave period ($=$ the period of the wavemaker oscillation).

We consider a generic, two-dimensional wavemaker configuration $\xi(z/h)$, described in Cartesian coordinates, which generates two-dimensional, irrotational motion of an inviscid, incompressible fluid in a semi-infinite channel of constant, still water depth h. The fluid motion is obtained from the negative gradient of a scalar velocity potential $\Phi(x, z, t)$ according to

$$[u, w] = -\nabla\Phi, \tag{1}$$

where $\nabla(\cdot) = [\partial/\partial x, \partial/\partial z]$.

The scalar velocity potential $\Phi(x, z, t)$ is a solution to

$$\nabla^2\Phi = 0; \quad x \geqslant \epsilon\left(\frac{S^*}{A^*}\right)\chi(z,t); \quad -h \leqslant z \leqslant \epsilon\eta(x,t), \tag{2a}$$

with boundary conditions

$$\frac{\partial\Phi}{\partial z} = 0; \quad x \geqslant \epsilon\left(\frac{S^*}{A^*}\right)\chi(-h,t); \quad z = -h, \tag{2b}$$

$$\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial\Phi}{\partial z} - \left[\epsilon\frac{\partial}{\partial t} - \tfrac{1}{2}\epsilon^2\nabla\Phi\cdot\nabla\right]|\nabla\Phi|^2 + \frac{\mathrm{d}B}{\mathrm{d}t} = 0; \quad x \geqslant \epsilon\left(\frac{S^*}{A^*}\right)\chi(\eta,t), \quad z = \epsilon\eta(x,t), \tag{2c}$$

$$\frac{\partial\Phi}{\partial x} + \left(\frac{S^*}{A^*}\right)\frac{\partial\chi}{\partial t} - \epsilon\left(\frac{S^*}{A^*}\right)\frac{\partial\Phi}{\partial z}\frac{\partial\chi}{\partial z} = 0; \quad x = \epsilon\left(\frac{S^*}{A^*}\right)\chi(z,t), \quad -h \leqslant z \leqslant \epsilon\eta(x,t), \tag{2d}$$

where $B(t)$ is the Bernoulli constant (Wehausen 1960; Mei 1983), which we use instead of $\bar{\eta}$ (Dean & Dalrymple 1984), and the small parameter $\epsilon = k^*A^* \ll 1$. In addition, a radiation condition is required as $x \to +\infty$ in order to ensure that propagating waves are only right-progressing or that the fluid velocities are bounded.

The displacement of the generic wavemaker is

$$\chi(z,t) = \xi(z/h)\sin(\omega_0 t + \gamma), \tag{3}$$

where $\xi(z/h)$ is a generic shape function.

For the double-articulated plane wavemaker of variable draught shown in figure 1, the generic shape function is

$$\xi(z/h) = \frac{h}{\Delta}\left[M\left(\frac{z}{h}\right) + B'\right]\left[U\left(\frac{z}{h} - \frac{d}{h} + 1\right) - U\left(\frac{z}{h} + \frac{b}{h}\right)\right] = \frac{h}{\Delta}\left[M\left(\frac{z}{h}\right) + B'\right]\delta U, \tag{4a}$$

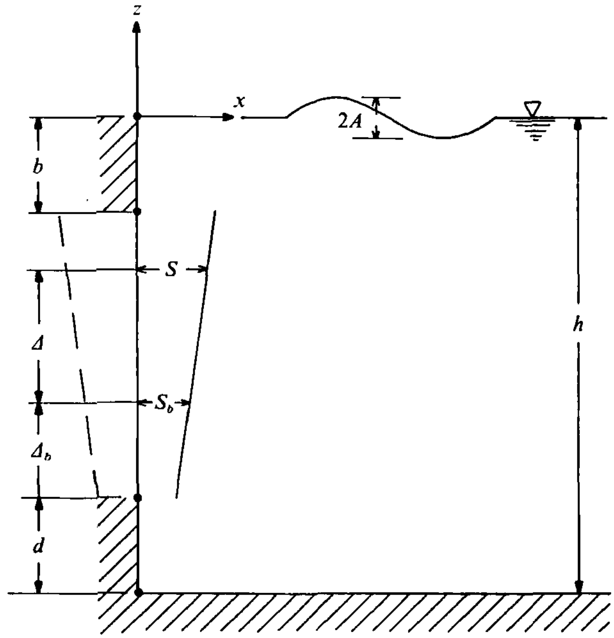


FIGURE 1. Definition sketch for a generic double-articulated planar wavemaker of variable draught.

where $U(\cdot)$ is the Heaviside step function and

$$M = 1-\frac{S_b^*}{S^*};\quad B' = \frac{\Delta}{h}-M\left(\frac{d}{h}+\frac{\Delta_b}{h}+\frac{\Delta}{h}-1\right), \qquad (4b,c)$$

where S^* is the dimensional wavemaker stroke measured at an arbitrary elevation $z/h = -1+(d/h)+(\Delta_b/h)+(\Delta/h)$ above the wave flume bottom. A piston wavemaker is represented by $S_b^*/S^* = 1$; and a wavemaker of full-depth draught is represented by $b/h = d/h = 0$.

The free-surface profile is

$$\eta(x,t) = \frac{\partial \Phi}{\partial t}-\tfrac{1}{2}\epsilon|\nabla\Phi|^2+B(t);\quad x \geqslant \epsilon\left(\frac{S^*}{A^*}\right)\chi(\eta,t),\quad z = \epsilon\eta(x,t), \qquad (5)$$

and the total pressure

$$p(x,z,t) = \frac{\partial \Phi}{\partial t}-\tfrac{1}{2}\epsilon|\nabla\Phi|^2-\frac{z}{\epsilon}+B(t);\quad x \geqslant \epsilon\left(\frac{S^*}{A^*}\right)\chi(z,t),\quad -h \leqslant z \leqslant \epsilon\eta(x,t). \qquad (6)$$

The free-surface conditions and the wavemaker kinematic boundary condition may be expanded in a Maclaurin as

$$\sum_{n=0}^{\infty}\frac{(\epsilon\eta)^n}{n!}\frac{\partial^n}{\partial z^n}\left[\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial\Phi}{\partial z}-\left(\epsilon\frac{\partial}{\partial t}-\tfrac{1}{2}\epsilon^2\nabla\Phi\cdot\nabla\right)|\nabla\Phi|^2+\frac{\mathrm{d}B}{\mathrm{d}t}\right] = 0;\quad x \geqslant 0,\quad z = 0, \qquad (7a)$$

$$\eta(x,t)-\sum_{n=0}^{\infty}\frac{(\epsilon\eta)^n}{n!}\frac{\partial^n}{\partial z^n}\left[\frac{\partial\Phi}{\partial t}-\tfrac{1}{2}\epsilon|\nabla\Phi|^2+B(t)\right] = 0;\quad x \geqslant 0,\quad z = 0, \qquad (7b)$$

$$\sum_{n=0}^{\infty}\frac{[\epsilon(S^*/A^*)\chi]^n}{n!}\frac{\partial^n}{\partial x^n}\left[\frac{\partial\Phi}{\partial x}+\left(\frac{S^*}{A^*}\right)\frac{\partial\chi}{\partial t}-\epsilon\left(\frac{S^*}{A^*}\right)\frac{\partial\Phi}{\partial z}\frac{\partial\chi}{\partial z}\right] = 0;\quad x = 0,\quad -h \leqslant z \leqslant 0. \qquad (7c)$$

In addition, the following variables are expanded in the small parameter ϵ:

$$\Phi(x,z,t) = \sum_{n=0}^{\infty} \epsilon^n \, {}_{n+1}\Phi(x,z,t), \tag{8a}$$

$$\eta(x,t) = \sum_{n=0}^{\infty} \epsilon^n \, {}_{n+1}\eta(x,t), \tag{8b}$$

$$B(t) = \sum_{n=0}^{\infty} \epsilon^n \, {}_{n+1}B(t), \tag{8c}$$

$$p(x,z,t) = p_s(z) + \sum_{n=0}^{\infty} \epsilon^n \, {}_{n+1}P(x,z,t), \tag{8d}$$

$$\omega t = \left(\sum_{n=0}^{\infty} \epsilon^n \omega_n\right) t = \tau, \tag{8e}$$

where $p_s(z) = z/\epsilon$ is the dimensionless hydrostatic pressure.

The Lindstedt–Poincaré perturbation of the frequency by (8e) leads to the following change of variables:

$$\frac{\partial}{\partial t} = \left(\sum_{n=0}^{\infty} \epsilon^n \omega_n\right) \frac{\partial}{\partial \tau} \tag{8f}$$

and, correspondingly, a free-surface operator defined by

$$\mathscr{L}_n(\cdot) = \left\{\left(\sum_{n=0}^{\infty} \epsilon^n \omega_n\right)^2 \frac{\partial^2}{\partial \tau^2} + \frac{\partial}{\partial z}\right\}(\cdot). \tag{8g}$$

2.1. ϵ^0 *solution*

The linear boundary value problem is

$$\nabla^2 \, {}_1\Phi = 0; \quad x \geqslant 0, \quad -h \leqslant z \leqslant 0, \tag{9a}$$

$$\frac{\partial \, {}_1\Phi}{\partial z} = 0; \quad x \geqslant 0, \quad z = -h, \tag{9b}$$

$$\mathscr{L}_0\{{}_1\Phi\} + \omega_0 \frac{\partial \, {}_1B}{\partial \tau} = 0; \quad x \geqslant 0, \quad z = 0, \tag{9c}$$

$$\frac{\partial \, {}_1\Phi}{\partial x} = -\omega_0 \left(\frac{S^*}{A^*}\right) \frac{\partial \chi}{\partial \tau}; \quad x = 0, \quad -h \leqslant z \leqslant 0, \tag{9d}$$

where
$$\mathscr{L}_0(\cdot) = \left\{\omega_0^2 \frac{\partial^2}{\partial \tau^2} + \frac{\partial}{\partial z}\right\}(\cdot). \tag{10}$$

A radiation condition is required as $x \to +\infty$ that will admit only right-progressing waves or bounded evanescent eigenmodes.

The first-order free-surface elevation and the dynamic pressure may be determined from

$${}_1\eta(x,\tau) = \omega_0 \frac{\partial \, {}_1\Phi}{\partial \tau} + {}_1B(\tau); \quad x \geqslant 0, \quad z = 0, \tag{11}$$

$${}_1P(x,z,\tau) = \omega_0 \frac{\partial \, {}_1\Phi}{\partial \tau} + {}_1B(\tau); \quad x \geqslant 0, \quad -h \leqslant z \leqslant 0. \tag{12}$$

The linear solution that is simple-harmonic is

$$ {}_1\Phi(x,z,\tau) = -a_1\,\phi_1\left(\frac{z}{h}\right)\sin(x-\tau-\gamma) - \cos(\tau+\gamma)\sum_{m=0} a_m\,\phi_m\left(\frac{z}{h}\right)\exp(-\alpha_m x), \quad (13) $$

where $$ \phi_m\left(\frac{z}{h}\right) = \frac{\cos[\alpha_m h(1+z/h)]}{n_m}, \quad (14a) $$

$$ n_m^2 = \int_{-1}^{0} \cos^2\left[\alpha_m h\left(1+\frac{z}{h}\right)\right] d\left(\frac{z}{h}\right) = \frac{2\alpha_m h + \sin 2\alpha_m h}{4\alpha_m h} \quad (14b) $$

provided that ${}_1B$ is identically zero and that

$$ \omega_0^2 h + \alpha_m h \tan\alpha_m h = 0, \quad (14c) $$

and that $\alpha_1 = +\mathrm{i} = +\surd(-1)$ in (14).

The coefficients a_m for the generic shape function for a planar wavemaker given by ($4a$) are

$$ a_1 = \frac{\omega_0 h}{h^3}\left(\frac{S^*}{A^*}\right)\left(\frac{h}{\Delta}\right) D_1\left(h, \frac{d}{h}, \frac{b}{h}, M, B'\right), \quad (15a) $$

$$ a_m = \frac{\omega_0 h}{(\alpha_m h)^3}\left(\frac{S^*}{A^*}\right)\left(\frac{h}{\Delta}\right) D_m\left(\alpha_m h, \frac{d}{h}, \frac{b}{h}, M, B'\right); \quad m \geqslant 2, \quad (15b) $$

where

$$ \begin{aligned} D_1\left(h, \frac{d}{h}, \frac{b}{h}, M, B'\right) = {} & B'h\left\{\phi_1(0)\cosh(b_u)[\tanh(h) - \tanh(b_u)] - \phi_1'\left(\frac{d_u}{h} - 1\right)\right\} \\ & + M\Big\{\phi_1(0)\cosh(b_u)[b_u(\tanh(b_u) - \tanh(h)) \\ & - 1 + \tanh(h)\tanh(b_u)] \\ & + \phi_1\left(\frac{d_u}{h} - 1\right)\left[1 + h\left(1 - \frac{d_u}{h}\right)\tanh(d_u)\right]\Big\}, \end{aligned} \quad (15c) $$

$$ \begin{aligned} D_m\left(\alpha_m h, \frac{d}{h}, \frac{b}{h}, M, B'\right) = {} & B'\alpha_m h\Big\{\phi_m(0)\cos(\alpha_m b_u)[\tan(\alpha_m b_u) - \tan(\alpha_m h)] \\ & + \phi_m'\left(\frac{d_u}{h} - 1\right)\Big\} + M\Big\{\phi_m(0)\cos(\alpha_m b_u)[\alpha_m b_u(\tan(\alpha_m h) \\ & - \tan(\alpha_m b_u)) - 1 - \tan(\alpha_m h)\tan(\alpha_m b_u)] \\ & + \phi_m\left(\frac{d_u}{h} - 1\right)\left[1 - \alpha_m h\left(1 - \frac{d_u}{h}\right)\tan(\alpha_m d_u)\right]\Big\}; \quad m \geqslant 2, \end{aligned} \quad (15d) $$

$$ b_u = b\,U(b/h); \quad d_u = d\,U(d/h), \quad (15e,f) $$

$$ \phi_1'(\cdot) = \frac{\sinh h(1+(\cdot))}{n_1}; \quad \phi_m'(\cdot) = \frac{\sin\alpha_m h(1+(\cdot))}{n_m}; \quad m \geqslant 2, \quad (15g,h) $$

where $U(\cdot)$ is the Heaviside step function which is required in order to ensure that negative values of the dimensions b or d are not used in the arguments of the transcendental functions in ($15c, d$). Recall that a piston wavemaker may be recovered from (15) when $S_b^*/S^* = 1$ and a wavemaker of full-depth draught when $b/h = d/h = 0$.

For generic planar wavemakers, the evanescent eigenseries in (13) converges at least as fast as $a_m \Rightarrow (\alpha_m h)^{-2} \sim [(m-1)\pi]^{-2}$. However, when $b/h = 0$, the convergence improves to $a_m \Rightarrow (\alpha_m h)^{-3} \sim [(m-1)\pi]^{-3}$ for a piston wavemaker of full-depth draught and for a hinged wavemaker of variable draught if $M^{-1} \equiv 1 + \Delta_b/\Delta$. This difference in the convergence of the first-order eigenseries will become important in estimating the convergence of second-order series; and will limit the geometries of planar wavemakers which can be solved using Stokes expansions.

Far away from the wavemaker ($x/h > 3$, say) the free-surface profile is

$$ {}_1\eta(x,\tau) = \cos(x-\tau-\gamma) \tag{16}$$

so that the dimensionless wavemaker gain function becomes

$$\left(\frac{S^*}{A^*}\right)\left(\frac{h}{\Delta}\right) = \left(\frac{h}{\omega_0}\right)^2 \left[\phi_1(0)\, D_1\left(h, \frac{b}{h}, \frac{d}{h}, M, B'\right)\right]^{-1}; \quad x > 3h, \tag{17}$$

which is valid for both hinged ($S_b^* \neq S^*$) and piston ($S_b^* = S^*$) wavemakers of variable draught.

Equation (17) reduces (15*a*, *b*) for the coefficients of the linear first-order potential to the following far-field expressions:

$$a_1 = [\omega_0\, \phi_1(0)]^{-1}, \tag{18a}$$

$$a_m = a_1 \left(\frac{h}{\alpha_m h}\right)^3 \left[\frac{D_m(\alpha_m h, b/h, d/h, M, B')}{D_1(h, b/h, d/h, M, B')}\right]; \quad m \geqslant 2. \tag{18b}$$

Some of the inner products required for the second-order ϵ solution may be simplified by substituting (3) and (13) into (9*d*) and noting that

$$\xi\left(\frac{z}{h}\right) = \omega_0^{-1}\left(\frac{A^*}{S^*}\right)\left\{a_1\,\phi_1(z/h) - \sum_{m=2} a_m\, \alpha_m\, \phi_m\left(\frac{z}{h}\right)\right\}, \tag{18c}$$

$$\frac{\partial \xi(z/h)}{\partial(z/h)} = \left(\frac{h}{\omega_0}\right)\left(\frac{A^*}{S^*}\right)\left\{a_1\,\phi_1'\left(\frac{z}{h}\right) + \sum_{m=2} a_m\, \alpha_m^2\, \phi_m'\left(\frac{z}{h}\right)\right\}. \tag{18d}$$

Equations (18) are valid for both piston ($M = 0$) and hinged ($M \neq 0$) wavemakers of variable draught.

2.2. ϵ^1 *solution*

The boundary value problem correct to second-order in ϵ is

$$\nabla^2\, {}_2\Phi = 0; \quad x \geqslant 0, \quad -h \leqslant z \leqslant 0, \tag{19a}$$

$$\frac{\partial\, {}_2\Phi}{\partial z} = 0; \quad x \geqslant 0, \quad z = -h, \tag{19b}$$

$$\mathscr{L}_0\{{}_2\Phi\} + \omega_0 \frac{\partial\, {}_2B}{\partial \tau} = -2\omega_0\,\omega_1 \frac{\partial^2\, {}_1\Phi}{\partial \tau^2} + \omega_0 \frac{\partial}{\partial \tau}|\nabla\, {}_1\Phi|^2 - {}_1\eta \frac{\partial}{\partial z}\mathscr{L}_0\{{}_1\Phi\}; \quad x \geqslant 0, \quad z = 0 \tag{19c}$$

$$\begin{aligned}\frac{\partial\, {}_2\Phi}{\partial x} &= \left(\frac{S^*}{A^*}\right)\left\{-\omega_1 \frac{\partial \chi}{\partial \tau} + \frac{\partial\, {}_1\Phi}{\partial z}\frac{\partial \chi}{\partial z} - \frac{\partial^2\, {}_1\Phi}{\partial x^2}\chi\right\} \\ &= -\omega_1\left(\frac{S^*}{A^*}\right)\frac{\partial \chi}{\partial \tau} + \frac{\partial}{\partial z}\left[\frac{\partial\, {}_1\Phi}{\partial z}\left(\frac{S^*}{A^*}\right)\chi\right]; \quad x = 0, \quad -h \leqslant z \leqslant 0. \end{aligned}\tag{19d}$$

The vertical derivative in the second term on the right-hand side of (19*d*) is in a form that is especially convenient for employing the first-order orthonormal eigenfunctions in (18*c*).

The solution to (19) must also satisfy a radiation condition as $x \to \infty$ that will admit only right-progressing waves or bounded fluid velocities. Because (19*d*) is an inhomogeneous Neumann condition, any constant times x may also be used for any time-independent solution for ${}_2\Phi$.

The free-surface elevation and dynamic pressure are

$$ {}_2\eta(x,\tau) = \omega_0 \frac{\partial\, {}_2\Phi}{\partial\tau} + \omega_1 \frac{\partial\, {}_1\Phi}{\partial\tau} - \tfrac{1}{2}|\nabla\, {}_1\Phi|^2 + \omega_0\, {}_1\eta \frac{\partial^2\, {}_1\Phi}{\partial z\, \partial\tau} + {}_2B(\tau); \quad x \geqslant 0, \quad z = 0. \tag{20} $$

$$ {}_2P(x,z,\tau) = \omega_0 \frac{\partial\, {}_2\Phi}{\partial\tau} + \omega_1 \frac{\partial\, {}_1\Phi}{\partial\tau} - \tfrac{1}{2}|\nabla\, {}_1\Phi|^2 + {}_2B(\tau); \quad x \geqslant 0, \quad -h \leqslant z \leqslant 0 \tag{21} $$

where for a zero-mean free-surface elevation,

$$ {}_2B = \left(\frac{a_1}{2n_1}\right)^2 \tag{22} $$

and $\partial\, {}_2B/\partial\tau = 0$ in (19*c*). In addition, the first term on the right-hand side of (19*c*) must vanish since $\partial^2\, {}_1\Phi/\partial\tau^2$ is a homogeneous solution of the linear operator $\mathscr{L}_0(\cdot)$ on the left-hand side of (19*c*) and would introduce a secular term of the form $\tau\, \partial\, {}_1\Phi/\partial\tau$ in the solution for ${}_2\Phi$. Since $\omega_0 > 0$, it is required that

$$ \omega_1 = 0, \tag{23} $$

which implies that the propagating wavenumber k^* is a constant correct to second-order ϵ. Consequently, the linear dispersion equations (14*c*) may be used to reduce some transcendental expressions at second-order in ϵ. However, the expression which results from the use of (14*c*) would not be valid for higher-order approximations.

It is customary in well-posed boundary-value problems with inhomogeneous boundary conditions on orthogonal boundaries such as those given by (19*c*, *d*) to linearly decompose the solution into complementary homogeneous and inhomogeneous solutions. Accordingly, the solution to (19) may be expressed as a linear combination of four scalar velocity potentials given by

$$ {}_2\Phi = {}_2\Phi^{\mathrm{S}} + {}_2\Phi^{\mathrm{e}} + {}_2\Phi^{\mathrm{f}} + {}_2\Psi, \tag{24} $$

in which ${}_2\Phi^{\mathrm{S}}$ is the second-order Stokes wave potential that is independent of the wavemaker motion (cf. Flick & Guza 1980); ${}_2\Phi^{\mathrm{e}}$ is a evanescent interaction potential; ${}_2\Phi^{\mathrm{f}}$ is a wavemaker-forced potential; and ${}_2\Psi$ is a time-independent potential that is required in order to satisfy (19*c*, *d*). The inhomogeneous boundary conditions in (19*c*, *d*) become

$$ \begin{aligned} \mathscr{L}_0\{{}_2\Phi^{\mathrm{S}} + {}_2\Phi^{\mathrm{e}} + {}_2\Phi^{\mathrm{f}} + {}_2\Psi\} = {} & a_1^2 f_1(\phi_1) \sin 2(x-\tau-\gamma) \\ & - a_1 \sin(x - 2\tau - 2\gamma) \sum_{m=2} a_m \exp(-\alpha_m x) f_2(\phi_1, \phi_m) \\ & - a_1 \cos(x - 2\tau - 2\gamma) \sum_{m=2} a_m \exp(-\alpha_m x) f_3(\phi_1, \phi_m) \\ & - \sin 2(\tau+\gamma) \sum_{m=2} \sum_{n=2} a_m a_n \exp[-(\alpha_m + \alpha_n) x] f_4(\phi_m, \phi_n) \\ & - a_1 \cos x \sum_{m=2} a_m \exp(-\alpha_m x) f_5(\phi_1, \phi_m); \quad x \geqslant 0, \quad z = 0 \end{aligned} \tag{25a} $$

and

$$\frac{\partial}{\partial x}\{{}_2\phi^{\mathrm{S}}+{}_2\Phi^{\mathrm{e}}+{}_2\Phi^{\mathrm{f}}+{}_2\Psi\} = \frac{a_1}{2h}W_1\left(\phi_1,\xi,\frac{z}{h}\right)[1-\cos 2(\tau+\gamma)]$$

$$+\frac{\sin 2(\tau+\gamma)}{2h}\sum_{m=2} a_m\alpha_m W_2\left(\phi_m,\xi,\frac{z}{h}\right);\quad x=0,\quad -h\leqslant z\leqslant 0,\quad (25b)$$

in which the nonlinear, free-surface interaction terms f_1, f_2, f_3, f_4 and f_5 and the nonlinear wavemaker interaction terms W_1 and W_2 are defined in the Appendix. The time-independent forcing terms given by $f_5(\phi_1,\phi_m)$ and by $W_1(\phi_1,\xi,z/h)$ appear to have been completely neglected in previously published solutions; while the evanescent interaction forcing terms $f_2(\phi_1,\phi_m)$ and $f_3(\phi_1,\phi_m)$ appear to have been only partially considered.

The second-order Stokes wave potential ${}_2\Phi^{\mathrm{S}}$ satisfies (19 a, b), a radiation condition, and

$$\mathscr{L}_0\{{}_2\Phi^{\mathrm{S}}\}-a_1^2 f_1(\phi_1)\sin 2(x-\tau-\gamma)=0;\quad x\geqslant 0,\quad z=0 \quad (26)$$

and is (Stokes 1847)

$${}_2\Phi^{\mathrm{S}} = -\left(\frac{a_1^2 f_1(\phi_1)}{2(2\omega_0^2-\tanh 2h)\cosh 2h}\right)\cosh 2h\left(1+\frac{z}{h}\right)\sin 2(x-\tau-\gamma)$$

$$= -\tfrac{3}{8}\omega_0\,\mathrm{cosech}^4 h\cosh 2h\left(1+\frac{z}{h}\right)\sin 2(x-\tau-\gamma) \quad (27)$$

correct to second-order in ϵ.

The evanescent interaction potential ${}_2\Phi^{\mathrm{e}}$ must satisfy (19 a, b), a radiation condition, and

$$\mathscr{L}_0\{{}_2\Phi^{\mathrm{e}}\}+a_1\sin(x-2\tau-2\gamma)\sum_{m=2}a_m\exp(-\alpha_m x)f_2(\phi_1,\phi_m)$$

$$+a_1\cos(x-2\tau-2\gamma)\sum_{m=2}a_m\exp(-\alpha_m x)f_3(\phi_1,\phi_m)$$

$$+\sin 2(\tau+\gamma)\sum_{m=2}\sum_{n=2}a_m a_n\exp[-(\alpha_m+\alpha_n)x]f_4(\phi_m,\phi_n)=0;$$

$$x\geqslant 0,\quad z=0 \quad (28)$$

and is assumed to be given by

$${}_2\Phi^{\mathrm{e}}(x,z,\tau) = a_1\cos(x-2\tau-2\gamma)\sum_{m=2}a_m\exp(-\alpha_m x)$$

$$\times\left[A_m\phi_1\left(\frac{z}{h}\right)\phi_m\left(\frac{z}{h}\right)+B_m\phi_1'\left(\frac{z}{h}\right)\phi_m'\left(\frac{z}{h}\right)\right]-a_1\sin(x-2\tau-2\gamma)$$

$$\times\sum_{m=2}a_m\exp(-\alpha_m x)\left[A_m\phi_1'\left(\frac{z}{h}\right)\phi_m'\left(\frac{z}{h}\right)-B_m\phi_1\left(\frac{z}{h}\right)\phi_m\left(\frac{z}{h}\right)\right]$$

$$-\sin 2(\tau+\gamma)\sum_{m=2}\sum_{n=2}a_m a_n\exp[-(\alpha_m+\alpha_n)x]C_{mn}$$

$$\times\left[\phi_m\left(\frac{z}{h}\right)\phi_n\left(\frac{z}{h}\right)-\phi_m'\left(\frac{z}{h}\right)\phi_n'\left(\frac{z}{h}\right)\right], \quad (29)$$

where $A_m = -\tfrac{1}{2}\omega_0\{[\omega_0^2(\tanh h-\alpha_m\tan\alpha_m h)-4\alpha_m\tanh h\tan\alpha_m h+\alpha_m^2-1]$
$\times(\tanh h-\alpha_m\tan\alpha_m h-4\omega_0^2)+16\omega_0^2\alpha_m\tanh h\tan\alpha_m h$

$$-4(\alpha_m \tan\alpha_m h + \alpha_m^2 \tanh h)\}\,[\mathrm{DET}^{-1}(A_m, B_m)]$$

$$= \frac{\alpha_m^2[3(4\omega_0^4+\alpha_m^2-1)+2\omega_0^4]}{\omega_0[(4\omega_0^4+\alpha_m^2-1)^2+(2\alpha_m)^2]}, \tag{30a}$$

$$B_m = -\tfrac{1}{2}\omega_0\{[\omega_0^2(\tanh h - \alpha_m \tan\alpha_m h) - 4\,\alpha_m \tanh h \tan\alpha_m h + \alpha_m^2 - 1]$$

$$\times (\alpha_m \tanh h + \tan\alpha_m h - 4\omega_0^2 \tanh h \tan\alpha_m h)$$

$$-4\alpha_m(4w_0^2 + \alpha_m \tan\alpha_m h - \tanh h)\}\,[\mathrm{DET}^{-1}(A_m, B_m)]$$

$$= -\frac{\alpha_m}{2\omega_0}\frac{[(4\omega_0^4+\alpha_m^2-1)(6\omega_0^4+\alpha_m^2-1)-8\alpha_m^2]}{[(4\omega_0^4+\alpha_m^2-1)^2+(2\alpha_m)^2]}, \tag{30b}$$

$$C_{mn} = -\tfrac{1}{4}\omega_0$$

$$\times\left[\frac{(\alpha_m+\alpha_n)^2+2\alpha_m\alpha_n-\omega_0^2(\alpha_m\tan\alpha_m h+\alpha_n\tan\alpha_n h)+4\alpha_m\alpha_n\tan\alpha_m h\tan\alpha_n h}{4\omega_0^2(1-\tan\alpha_m h\tan\alpha_n h)+(\alpha_m+\alpha_n)(\tan\alpha_m h+\tan\alpha_n h)}\right]$$

$$= \frac{\alpha_m\alpha_n}{4\omega_0}\frac{[(\alpha_m+\alpha_n)^2+2\alpha_m\alpha_n+6\omega_0^4]}{[(2\omega_0^2)^2+(\alpha_m-\alpha_n)^2]}, \tag{30c}$$

$$\mathrm{DET}(A_m, B_m) = 8\omega_0^2[(2\omega_0^2\tanh h\tan\alpha_m h - \tan\alpha_m h - \alpha_m\tanh h)\tanh h\tan\alpha_m h$$

$$+2\omega_0^2 - \tanh h + \alpha_m\tan\alpha_m h] + (\alpha_m^2+1)(\tan^2\alpha_m h + \tanh^2 h)$$

$$= \left(\frac{\omega_0^2}{\alpha_m}\right)^2[(4\omega_0^4+\alpha_m^2-1)^2+(2\alpha_m)^2], \tag{30d}$$

correct to second-order in ϵ.

For generic planar wavemakers where the first-order evanescent eigenseries converges only as fast as $a_m \Rightarrow (\alpha_m h)^{-2} \sim [(m-1)\pi]^{-2}$, the single summation series in (29) will converge at least as fast as $a_m B_m \propto (\alpha_m h)^{-1}$; and the double summation series in (29) will converge at least as fast as $a_n C_{mn} \propto (\alpha_n h)^{-1}$. This limits the practical application of the evanescent interaction potential ${}_2\Phi^{e}$ to those planar wavemaker geometries where the convergence of the first-order evanescent eigenseries improves to $a_m \Rightarrow (\alpha_m h)^{-3} \sim [(m-1)\pi]^{-3}$.

The wavemaker-forced potential ${}_2\Phi^{f}$ must satisfy (19a, b), a homogeneous form of (10), a radiation condition, and

$$\frac{\partial}{\partial x}\{{}_2\Phi^{f}\} + \frac{\partial}{\partial x}\{{}_2\Phi^{S} + {}_2\Phi^{e}\} + \frac{\cos 2(\tau+\gamma)}{2h}a_1 W_1\left(\phi_1, \xi, \frac{z}{h}\right)$$

$$-\frac{\sin 2(\tau+\gamma)}{2h}\sum_{m=2} a_m \alpha_m W_2\left(\phi_m, \xi, \frac{z}{h}\right) = 0; \quad x=0, \quad -h \leqslant z \leqslant 0 \tag{31}$$

and is assumed to be given by

$${}_2\Phi^{f}(x,z,\tau) = [E_1\cos(\beta_1 x - 2(\tau+\gamma)) + F_1\sin(\beta_1 x - 2(\tau+\gamma))]\,Q_1\left(\frac{z}{h}\right)$$

$$-\sum_{j=2}\exp(-\beta_j x)\,[E_j\sin 2(\tau+\gamma) + F_j\cos 2(\tau+\gamma)]\,Q_j\left(\frac{z}{h}\right), \tag{32}$$

where
$$Q_j\left(\frac{z}{h}\right) = \frac{\cos\beta_j h(1+z/h)}{N_j}; \quad j \geqslant 1, \tag{33a}$$

$$N_j^2 = \int_{-1}^{0}\cos^2\left[\beta_j h\left(1+\frac{z}{h}\right)\right]\mathrm{d}\left(\frac{z}{h}\right) = \frac{(2\beta_j h + \sin 2\beta_j h)}{4\beta_j h}; \quad j \geqslant 1, \tag{33b}$$

provided that $$4\omega_0^2 h + \beta_j h \tan \beta_j h = 0; \quad j \geqslant 1 \tag{33c}$$

and that $\beta_1 = \mathrm{i}\beta_1$ in (33).

The coefficients E_j and F_j may be computed from

$$|\beta_j| E_j = -\frac{a_1}{h} \sum_{m=2} a_m \left[A_m \langle (\phi_1' \phi_m)', Q_j \rangle_{z/h} + \left(\frac{B_m}{\alpha_m} - \frac{1}{2\omega_0} \right) \langle \alpha_m (\phi_1 \phi_m')', Q_j \rangle_{z/h} \right]$$

$$- \sum_{m=2} \sum_{n=2} \frac{a_m a_n}{h} \left(C_{mn} + \frac{\alpha_m \alpha_n}{4\omega_0} \right) [\langle (\phi_m \phi_n')', Q_j \rangle_{z/h} + \langle (\phi_m' \phi_n)', Q_j \rangle_{z/h}], \tag{34a}$$

$$|\beta_j| F_j = \left(\frac{a_1}{2\omega_0^2} \right)^2 \left(\frac{3 - 5\omega_0^4}{\omega_0 h} \right) \langle (\phi_1 \phi_1')', Q_j \rangle_{z/h}$$

$$+ \frac{a_1}{h} \sum_{m=2} a_m \left[A_m \langle (\phi_1 \phi_m')', Q_j \rangle_{z/h} - \left(B_m - \frac{\alpha_m}{2\omega_0} \right) \langle (\phi_1' \phi_m)', Q_j \rangle_{z/h} \right], \tag{34b}$$

where $(\cdot\cdot)' = \partial/\partial(z/h)(\cdot\cdot)$ and where the identity

$$\sum_{m=2} \sum_{n=2} \alpha_m^2 \alpha_n = \frac{1}{2} \sum_{m=2} \sum_{n=2} (\alpha_m + \alpha_n) \alpha_m \alpha_n \tag{35}$$

has been used to make the double-sum term derived from $W_2(\phi_m, \xi, z/h)$ in (34a) symmetric. Substituting (30) and (A 8) into (34) and persevering through some very tedious algebra eventually gives

$$|\beta_j| E_j = \left[\frac{a_1 \omega_0}{h} \right] \phi_1(0)\, Q_j(0) \sum_{m=2} a_m \phi_m(0) \frac{[2\beta_j^2 \alpha_m^2 + (\beta_j^2 + \alpha_m^2 + 1)(5\omega_0^4 - 1)]}{[(\alpha_m^2 - 1 - \beta_j^2)^2 + (2\alpha_m)^2]}$$

$$+ \left[\frac{\omega_0}{2h} \right] Q_j(0) \sum_{m=2} \sum_{n=2} a_m a_n (\alpha_m + \alpha_n) \phi_m(0)\, \phi_n(0) \frac{[5\omega_0^4 + \alpha_m^2 + \alpha_m \alpha_n + \alpha_n^2]}{[(\alpha_m + \alpha_n)^2 - \beta_j^2]}, \tag{36a}$$

$$|\beta_j| F_j = \left[\frac{a_1^2 \omega_0}{h} \right] \phi_1^2(0)\, Q_j(0) \left[\frac{5\omega_0^4 - 3}{4 + \beta_j^2} \right]$$

$$+ \left[\frac{a_1 \omega_0}{h} \right] \phi_1(0)\, Q_j(0) \sum_{m=2} a_m \alpha_m \phi_m(0) \left[\frac{2\beta_j^2 + (\alpha_m^2 + 1 - \beta_j^2)(5\omega_0^4 + \alpha_m^2)}{(\alpha_m^2 - 1 - \beta_j^2)^2 + (2\alpha_m)^2} \right], \tag{36b}$$

where $\beta_1 = \mathrm{i}\beta_1$.

For generic planar wavemakers where the first-order evanescent eigenseries converges only as fast as $a_m \Rightarrow (\alpha_m h)^{-2} \sim [(m-1)\pi]^{-2}$, the single summation series in (36a) will converge at least as fast as $a_m \alpha_m^{-2} \propto (\alpha_m h)^{-4} \sim [(m-1)]^{-4}$; and the double summation series in (36a) and the single summation series in (36b) will converge at least as fast as $a_m \alpha_m \propto (\alpha_m h)^{-1} \sim [(m-1)\pi]^{-1}$. The eigenseries for $Q_j(z/h)$ will converge at least as fast as $(\beta_j h)^{-3} \sim [(j-1)\pi]^{-3}$.

Note that the coefficients E_j and F_j in (36) depend on the first-order evanescent eigenseries; even for the propagating free-wave potential, $j = 1$. This implies that the amplitude of the second-order free-wave computed from ${}_2\Phi^{\mathrm{f}}$ depends on the first-order evanescent eigenseries which cannot be neglected at second-order as previously reported.

The dimensionless amplitude of the second-order free wave, a_2^{f} may be computed from

$$a_2^{\mathrm{f}} = 2\omega_0 Q_1(0)\, [E_1^2 + F_1^2]^{\frac{1}{2}} \tag{36c}$$

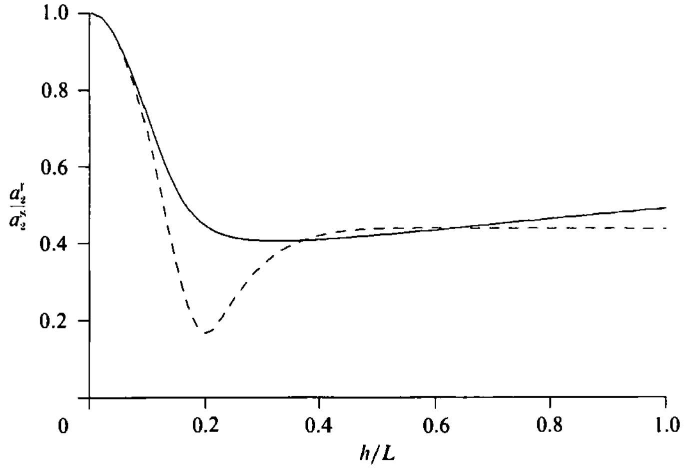


FIGURE 2. Dimensionless amplitude ratio a_2^r/a_2^s for a full-draught piston wavemaker ($b/h = d/h = 0$): ——, with the evanescent interaction potential; ---, without the evanescent interaction potential ($L = 2\pi$).

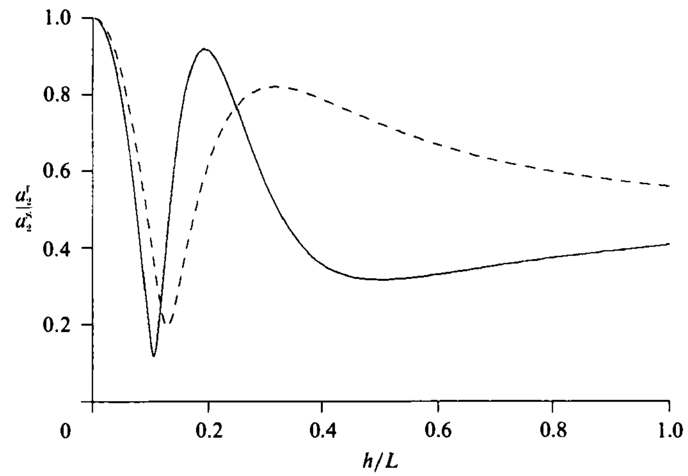


FIGURE 3. Dimensionless amplitude ratio a_2^r/a_2^s for a full-draught hinged wavemaker ($b/h = d/h = 0$): ——, with the evanescent interaction potential; ---, without the evanescent interaction potential ($L = 2\pi$).

and the dimensionless amplitude of the second-order Stokes wave from

$$a_2^{\mathrm{S}} = \frac{\cosh h\,(\cosh 2h + 2)}{4 \sinh^3 h}. \tag{36d}$$

The ratio of a_2^r/a_2^s for a full-draught piston wavemaker is shown in figure 2 and for a full-draught hinged wavemaker in figure 3. These ratios are also compared with the amplitude ratios in which the evanescent interaction potential ${}_2\Phi^e$ in (31) was neglected.

2.3. *Time-independent solutions*

An interesting feature of the second-order problem which has not been previously given much detailed attention is the time-independent potential ${}_2\Psi(x, z)$ in (24). This time-independent solution is found to accurately estimate the mean return flow in a wave flume computed by the Eulerian method. This mean return current due to Stokes drift is usually estimated from a principle of kinematic conservation of mass flux.

The time-independent potential ${}_2\Psi(x, z)$ must satisfy

$$\nabla^2\, {}_2\Psi = 0; \quad x \geqslant 0, \quad -h \leqslant z \leqslant 0, \tag{37a}$$

$$\frac{\partial\, {}_2\Psi}{\partial z} = 0; \quad x \geqslant 0, \quad z = -h, \tag{37b}$$

$$\mathscr{L}_0\{{}_2\Psi\} \equiv \frac{\partial\, {}_2\Psi}{\partial z} = -a_1 \cos x \sum_{m-2} a_m \exp(-\alpha_m x) f_5(\phi_1, \phi_m); \quad x \geqslant 0, \quad z = 0, \tag{37c}$$

$$\frac{\partial\, {}_2\Psi}{\partial x} = \frac{a_1}{2h} W_1\left(\phi_m, \xi, \frac{z}{h}\right); \quad x = 0, \quad -h \leqslant z \leqslant 0. \tag{37d}$$

Since the time-independent solution is not a progressive wave, the radiation condition as $x \rightarrow +\infty$ may be relaxed to admit bounded time-independent velocities (Wehausen 1960).

Because (37 c, d) are inhomogeneous, ${}_2\Psi$ may also be decomposed into

$${}_2\Psi = {}_2\Psi^e + {}_2\Psi^f. \tag{38}$$

Following the procedure used to obtain the time-dependent evanescent interaction potential ${}_2\Phi^e$, a solution for the time-independent evanescent interaction potential ${}_2\Psi^e$ is assumed to be of the form

$$\begin{aligned} {}_2\Psi^e(x, z) = {} & a_1 \cos x \sum_{m-2} a_m \exp(-\alpha_m x)\left[b_m \phi_1\left(\frac{z}{h}\right)\phi_m\left(\frac{z}{h}\right) + c_m \phi_1'\left(\frac{z}{h}\right)\phi_m'\left(\frac{z}{h}\right)\right] \\ & -a_1 \sin x \sum_{m-2} a_m \exp(-\alpha_m x)\left[b_m \phi_1'\left(\frac{z}{h}\right)\phi_m'\left(\frac{z}{h}\right) - c_m \phi_1\left(\frac{z}{h}\right)\phi_m\left(\frac{z}{h}\right)\right], \end{aligned} \tag{39}$$

where

$$b_m = -\tfrac{1}{2}\omega_0 \frac{(\tanh h - \alpha_m \tan \alpha_m h)}{(\tanh^2 h + \tan^2 \alpha_m h)} = -\frac{\alpha_m^2}{\omega_0(\alpha_m^2 + 1)}, \tag{40a}$$

$$c_m = -\tfrac{1}{2}\omega_0 \frac{(\alpha_m \tanh h + \tan \alpha_m h)}{(\tanh^2 h + \tan^2 \alpha_m h)} = -\frac{\alpha_m(\alpha_m^2 - 1)}{2\omega_0(\alpha_m^2 + 1)}, \tag{40b}$$

correct to second-order in ϵ.

For generic planar wavemakers where the first-order evanescent eigenseries converges only as fast as $a_m \Rightarrow (\alpha_m h)^{-2} \sim [(m-1)\pi]^{-2}$, the single summation series in (39) will converge at least as fast as $a_m c_m \propto (\alpha_m h)^{-1} \sim [(m-1)\pi]^{-1}$. This limits the practical application of the time-independent evanescent interaction potential ${}_2\Psi^e$ to those planar wavemaker geometries where the convergence of the first-order evanescent eigenseries improves to $a_m \Rightarrow (\alpha_m h)^{-3} \sim [(m-1)\pi]^{-3}$.

The time-independent wavemaker-forced potential ${}_2\Psi^f$ must satisfy a homogeneous free-surface condition, and the time-independent terms in the nonlinear inhomogeneous wavemaker boundary condition in (37 d) given by

$$\frac{\partial\, {}_2\Psi^f}{\partial x} - \frac{a_1}{2h} W_1\left(\phi_1, \xi, \frac{z}{h}\right) + \frac{\partial\, {}_2\Psi^e}{\partial x} = 0; \quad x = 0, \quad -h \leqslant z \leqslant 0. \tag{41}$$

Following the procedure used to obtain the time-dependent wavemaker-forced wave potential ${}_2\Phi^f$, a solution for ${}_2\Psi^f$ is assumed to be given by the following eigenfunction expansion:

$$ {}_2\Psi^f(x,z) = \sum_{j=0} d_j \psi_j\left(\frac{z}{h}\right)[\delta_{j0}x + (1-\delta_{j0})\exp(-\mu_j x)] \tag{42}$$

where
$$\psi_j\left(\frac{z}{h}\right) = (2-\delta_{j0})^{\frac{1}{2}} \cos \mu_j h \left(1+\frac{z}{h}\right); \quad j \geqslant 0 \tag{43a}$$

provided that
$$\mu_j h = j\pi; \quad j \geqslant 0. \tag{43b}$$

The coefficients d_j may be computed from

$$d_j(\delta_{j0}-\mu_j) = \left[\frac{a_1^2}{2h\omega_0}\right]\langle(\phi_1\phi_1')', \psi_j\rangle_{z/h}$$
$$+\frac{a_1}{h}\sum_{m=2} a_m\left[b_m\langle(\phi_1\phi_m')', \psi_j\rangle_{z/h} - \left(\frac{\alpha_m}{2\omega_0}+c_m\right)\langle(\phi_1'\phi_m)', \psi_j\rangle_{z/h}\right]. \tag{44a}$$

Substituting (40) and (A 8) into (44a) and integrating gives

$$d_j = \frac{2(-1)^j(2-\delta_{j0})^{\frac{1}{2}}}{\omega_0(\delta_{j0}-\mu_j)h}\left[(4+\mu_j^2)^{-1} - \omega_0\mu_j^2 \sum_{m=2}\frac{a_m\alpha_m\phi_m(0)}{(\alpha_m^2-1-\mu_j^2)^2+(2\alpha_m)^2}\right]; \quad j \geqslant 0. \tag{44b}$$

For $j=0$, $d_0 h = (2\omega_0)^{-1}$ which is exactly equal to the magnitude of the Stokes drift! This is discussed in more detail in §3.

For generic planar wavemakers where the first-order evanescent eigenseries converges only as fast as $a_m \Rightarrow (\alpha_m h)^{-2} \sim [(m-1)\pi]^{-2}$, the series in (44$b$) for d_j will converge at least as fast as $a_m\alpha_m^{-3} \propto (\alpha_m h)^{-5} \sim [(m-1)\pi]^{-5}$. The eigenseries for $\psi_j(z/h)$ will converge at least as fast as $(\mu_j h)^{-3} = (j\pi)^{-3}$.

3. Stokes drift

It is of interest to evaluate the mean horizontal fluid momentum per unit area correct to second-order in ϵ. The time- and depth-averaged mean horizontal fluid momentum per unit area is defined by (Phillips 1977)

$$M_{\mathrm{E(L)}} = h\left\langle \int_{-1}^{\eta/h} U_{\mathrm{E(L)}}\,\mathrm{d}\left(\frac{z}{h}\right)\right\rangle_{2\pi} \tag{45}$$

where the dimensionless temporal averaging operator is defined by

$$\langle\cdot\rangle_{2\pi} = (2\pi)^{-1}\textstyle\int_0^{2\pi}(\cdot)\,\mathrm{d}\tau$$

and $U_{\mathrm{E(L)}}$ is an Eulerian (Lagrangian) horizontal velocity component.

3.1. *Eulerian*

The horizontal component of the Eulerian velocity determined from (1), (13), and (24), is approximately

$$M_{\mathrm{E}} = \bar{U}_{\Psi} + \bar{U}_{\Phi} + O(\epsilon^2). \tag{46}$$

The mean horizontal component $\bar{U}_{\Psi}$ that is forced by the time-independent boundary conditions $f_5(\phi_1, \phi_m)$ and $W_1(\phi_1, \xi, z/h)$ may be computed from ${}_2\Psi(x,z)$ according to

$$\bar{U}_{\Psi} = -\epsilon h \int_{-1}^{0} \frac{\partial\, {}_2\Psi}{\partial x}\,\mathrm{d}\left(\frac{z}{h}\right)$$
$$= \bar{U}_{\Psi,\mathrm{e}}(a_m, \alpha_m h, x) + \bar{U}_{\Psi,\infty}(d_0), \tag{47a}$$

where $$\bar{U}_{\Psi,\mathrm{e}}(a_m, \alpha_m h, x) = -\tfrac{1}{2}\epsilon \sum_{m-2} a_m \phi_m(0)\,[\sin x - \alpha_m \cos x] \exp(-\alpha_m x) \tag{47b}$$

and $$\bar{U}_{\Psi,\infty}(d_0) = -\epsilon d_0 h = -\epsilon(2\omega_0)^{-1}. \tag{47c}$$

Similarly, the mean horizontal component $\bar{U}_{\Phi}$ may be estimated from the first-order eigenmodes ${}_1\Phi(x, z, \tau)$ according to

$$\begin{aligned}\bar{U}_{\Phi} &= -\epsilon\omega_0 \left\langle \left(\frac{\partial\, {}_1\Phi}{\partial x}\right)\left(\frac{\partial\, {}_1\Phi}{\partial \tau}\right)\right\rangle_{2\pi}; \quad z = 0 \\ &= \bar{U}_{\Phi,\mathrm{e}}(a_m, \alpha_m h, x) + \bar{U}_{\Phi,\infty}(\omega_0),\end{aligned} \tag{48a}$$

where $$\bar{U}_{\Phi,\mathrm{e}}(a_m, \alpha_m h, x) = \tfrac{1}{2}\epsilon \sum_{m-2} a_m \phi_m(0)\,(\sin x - \alpha_m \cos x) \exp(-\alpha_m x) \tag{48b}$$

and $$\bar{U}_{\Phi,\infty}(\omega_0) = \epsilon(2\omega_0)^{-1}, \tag{48c}$$

which is equal in magnitude but opposite in direction to $\bar{U}_{\Psi,\infty}(d_0)$! We have the remarkable result that a component of the complete time-independent solution to the nonlinear boundary-value problem correct to second-order in ϵ accurately estimates the mean return current in a closed wave flume. Previous estimates of this mean return current have been computed by an alternative kinematic principle (Longuet-Higgins 1953).

3.2. *Lagrangian*

The dimensionless Lagrangian induced-streaming velocity may be estimated from the Eulerian velocity (Mei 1983) by, approximately,

$$\boldsymbol{U}_{\mathrm{L}} = -\epsilon \boldsymbol{\nabla}\, {}_2\Psi + \frac{\epsilon}{\omega_0}\left\langle \left[\left(\int^{\tau} \boldsymbol{\nabla}\, {}_1\Phi \,\mathrm{d}\tau'\right)\cdot \boldsymbol{\nabla}\right] \boldsymbol{\nabla}\, {}_1\Phi \right\rangle_{2\pi} + O(\epsilon^2), \tag{49}$$

where the dimensionless Lagrangian induced-streaming velocity is $\boldsymbol{U}_{\mathrm{L}} = [U_{\mathrm{L}}, W_{\mathrm{L}}]$. The mean horizontal component $U_{\mathrm{L}}(x, z/h)$ is, approximately,

$$\begin{aligned} U_{\mathrm{L}}\left(x, \frac{z}{h}\right) = -\epsilon \frac{\partial\, {}_2\Psi}{\partial x} + \frac{\epsilon}{\omega_0}\Bigg\{ \frac{\cosh 2h(1 + z/h)}{\sinh 2h} + \tfrac{1}{2} a_1 \sum_{m-2} a_m \alpha_m \exp(-\alpha_m x) \\ \times \left[\phi_1'\left(\frac{z}{h}\right)\phi_m'\left(\frac{z}{h}\right)(\alpha_m \cos x - \sin x) - \phi_1\left(\frac{z}{h}\right)\phi_m\left(\frac{z}{h}\right)(\alpha_m \sin x + \cos x)\right]\Bigg\}, \end{aligned} \tag{50a}$$

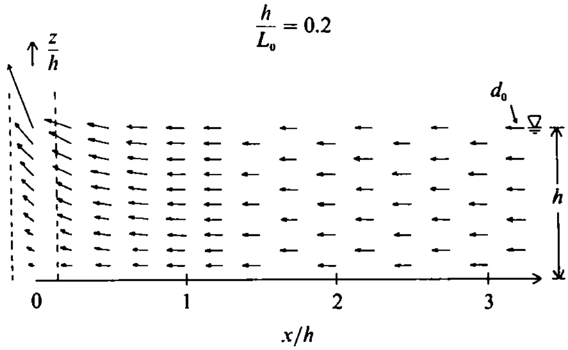
and the mean vertical component $W_{\mathrm{L}}(x, z/h)$ is, approximately,

$$\begin{aligned} W_{\mathrm{L}}\left(x, \frac{z}{h}\right) = -\epsilon \frac{\partial\, {}_2\Psi}{\partial z} - \left(\frac{\epsilon a_1}{2\omega_0}\right) \sum_{m-2} a_m \alpha_m \exp(-\alpha_m x) \\ \times \left[\phi_1\left(\frac{z}{h}\right)\phi_m'\left(\frac{z}{h}\right)(\alpha_m \sin x - \cos x) + \phi_1'\left(\frac{z}{h}\right)\phi_m\left(\frac{z}{h}\right)(\alpha_m \cos x + \sin x)\right]. \end{aligned} \tag{50b}$$

Substituting (50a) into (45) and integrating gives the mean horizontal momentum per unit area as, approximately,

$$M_{\mathrm{L}} = \bar{U}_{\Psi,\mathrm{e}}(a_m, \alpha_m h, x) + \bar{U}_{\Psi,\infty}(d_0) + \bar{U}_{\Phi,\mathrm{e}}(a_m, \alpha_m h, x) + \bar{U}_{\Phi,\infty}(\omega_0) + O(\epsilon^2), \tag{51}$$

which is identical to the mean horizontal momentum per unit area from the Eulerian description (Phillips (1977).



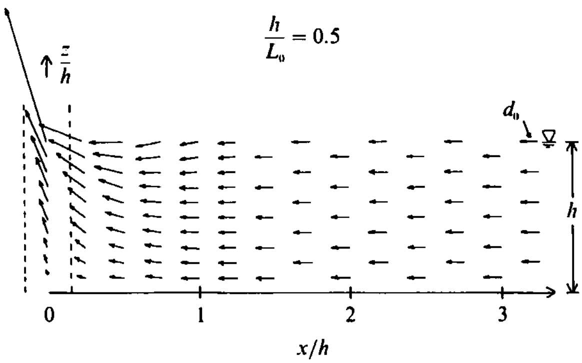


FIGURE 4. Spatial distribution of the second-order in ϵ time-independent velocity for a full-draught piston wavemaker ($L_0 = T^2/2\pi$).

3.3. *Time-independent Eulerian velocity*

The mean horizontal momentum per unit area computed from (47) and (48) and from (51) give, approximately,

$$M_{\mathrm{E(L)}} \sim \bar{U}_{\Psi} + \bar{U}_{\Phi} + O(\epsilon^2) \sim 0 \tag{52}$$

correct to second-order in ϵ.

Again, this result is because the component $\bar{U}_{\Psi}$ derived from the time-independent solution ${}_2\Psi$ is equal in magnitude but opposite in direction to the Stokes drift in both the near and far fields. The spatial distribution of the time-independent Eulerian velocities computed from the time-independent potential ${}_2\Psi$ reveal the importance of the contributions from the first-order evanescent eigenseries.

The horizontal component is

$$\begin{aligned} U_{\Psi}\left(x, \frac{z}{h}\right) &= -\epsilon \frac{\partial\, {}_2\Psi^{\mathrm{e}}}{\partial x} - \epsilon \frac{\partial\, {}_2\Psi^{\mathrm{t}}}{\partial x} \\ &= -\left(\frac{\epsilon a_1}{2\omega_0}\right) \sum_{m=2} a_m \alpha_m \left[\phi_1\left(\frac{z}{h}\right)\phi_m\left(\frac{z}{h}\right)(\cos x + \alpha_m \sin x)\right. \\ &\quad \left. + \phi_1'\left(\frac{z}{h}\right)\phi_m'\left(\frac{z}{h}\right)(\alpha_m \cos x - \sin x)\right] \exp(-\alpha_m x) \\ &\quad - \epsilon \sum_{j=0} d_j \psi_j\left(\frac{z}{h}\right)[\delta_{j0} - \mu_j \exp(-\mu_j x)] \end{aligned} \tag{53a}$$

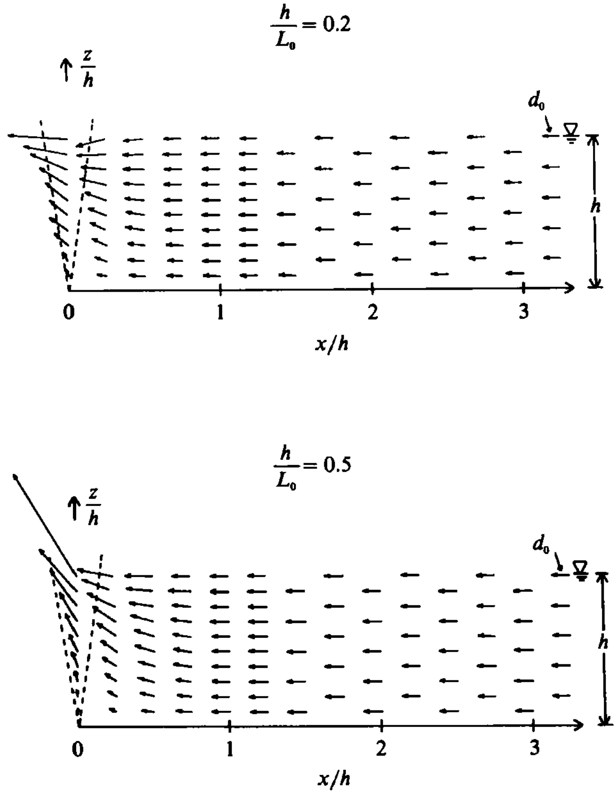


FIGURE 5. Spatial distribution of the second-order in ϵ time-independent velocity for a full-draught hinged wavemaker ($L_0 = T^2/2\pi$).

and the vertical component is

$$W_\psi\left(x, \frac{z}{h}\right) = -\epsilon \frac{\partial\, {}_2\Psi^e}{\partial z} - \epsilon \frac{\partial\, {}_2\Psi^t}{\partial z}$$

$$= -\left(\frac{\epsilon a_1}{2\omega_0}\right) \sum_{m=2} a_m \alpha_m \exp(-\alpha_m x) \left[\phi_1'\left(\frac{z}{h}\right) \phi_m\left(\frac{z}{h}\right)(\sin x - \alpha_m \cos x) + \phi_1\left(\frac{z}{h}\right) \phi_m'\left(\frac{z}{h}\right)(\cos x + \alpha_m \sin x)\right] + \epsilon \sum_{j=1} d_j \mu_j \psi_j'\left(\frac{z}{h}\right) \exp(-\mu_j x), \quad (53b)$$

where $\psi_j'(z/h) = (2-\delta_{j0})^{\frac{1}{2}} \sin \mu_j h(1+z/h)$. The series for both the horizontal and vertical components computed from ${}_2\Psi^e$ in (53) are seen to be non-converging for those planar wavemaker geometries which generate first-order eigenseries that converge only as fast as $a_m \propto (\alpha_m h)^{-2} \sim [(m-1)\pi]^{-2}$.

The magnitude of the time-independent velocity vector $\epsilon^{-1}[U_\psi^2(x, z/h) + W_\psi^2(x, z/h)]^{\frac{1}{2}}$ is illustrated in figures 4 and 5 for two relative water depths; namely $h/L_0 = 0.2$ (relatively shallow water); and $h/L_0 = 0.5$ (relatively deep water); where the deep-water wavelength $L_0 = T^2/2\pi$. Figures 4 and 5 illustrate the importance of the first-order evanescent eigenseries to the mean circulation pattern in a closed wave flume, especially near the wavemaker. The far-field velocity is equal to d_0 which was shown in (47*c*) to be proportional to the Stokes drift and to accurately estimate the magnitude of the mean return flow due to Stokes drift.

4. Summary

A complete nonlinear analytical solution that is correct to second-order in ϵ is given for the two-dimensional wave motion forced by a generic planar wavemaker. The planar wavemaker may be double articulated and includes both piston and hinged wavemakers of variable draught. The solution for the second-order potential is a linear combination of three time-dependent potentials and two time-independent potentials.

The first-order evanescent eigenseries contribute to the second-order solution in two important ways. The first way is that when computing the amplitude of the second-order free wave, it was shown in (36) that the first-order eigenseries cannot be neglected. The second way is that some of the second-order potentials which contain the first-order eigenseries are non-converging for those geometries of generic planar wavemakers that converge only as fast as $a_m \Rightarrow (\alpha_m h)^{-2} \sim [(m-1)\pi]^{-2}$. This gives the important result that second-order solutions to the nonlinear wavemaker boundary-value problem obtained by Stokes expansion will not converge for all planar wavemaker geometries.

When the wavemaker kinematic boundary condition is applied on the instantaneous boundary at $x = \chi(z/h, t)$ of a planar wavemaker, the time-independent forcing term $\propto \nabla\chi \cdot \nabla\Phi$ is non-zero only for a hinged wavemaker; but it becomes non-zero for both piston and hinged wavemakers when the kinematic boundary condition is expanded in a Maclaurin series about the equilibrium position at $x = 0$. The time-independent forcing on the free-surface boundary for both piston and hinged wavemakers is evanescent only in x. This time-independent solution has been neglected in previous Stokes expansions for the nonlinear wavemaker boundary-value problem.

The mean horizontal momentum per unit area computed from both the Eulerian and the Lagrangian methods is equal to zero, correct to second-order in ϵ. This is a result of the time-independent potentials generating mean horizontal velocities which are equal in magnitude but opposite in direction to the Stokes drift generated from both the propagating and evanescent first-order potentials. The complete solution to the weakly nonlinear wavemaker boundary-value problem accurately estimates the mean return current needed to maintain zero mass flux in a bounded domain. This estimate for the mean return current due to the Stokes drift generated by the propagating wave is usually estimated from a principle of kinematic conservation of mass flux.

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Appendix. Nonlinear interaction coefficients

The nonlinear, inhomogeneous free-surface interaction terms in (25a) are defined from first-order quantities by the following:

$$f_1(\phi_1) = \tfrac{1}{2}\omega_0[2\phi_1^2(-1)+\phi_1^2(0)(1-\omega_0^2\tanh h)] = \frac{3\omega_0}{2n_1^2}, \tag{A 1}$$

$$f_2(\phi_1,\phi_m) = 2\omega_0\alpha_m\phi_1(0)\phi_m(0), \tag{A 2}$$

$$\begin{aligned} f_3(\phi_1,\phi_m) &= \tfrac{1}{2}\omega_0\phi_1(0)\phi_m(0)[\omega_0^2(\tanh h-\alpha_m\tan\alpha_m h)-4\alpha_m\tanh h\tan\alpha_m h+\alpha_m^2-1] \\ &= \tfrac{1}{2}\omega_0\phi_1(0)\phi_m(0)[6\omega_0^4+\alpha_m^2-1], \end{aligned} \tag{A 3}$$

$$\begin{aligned} f_4(\phi_m,\phi_n) &= \tfrac{1}{4}\omega_0\phi_m(0)\phi_n(0)[(\alpha_m+\alpha_n)^2+2\alpha_m\alpha_n-\omega_0^2(\alpha_m\tan\alpha_m h \\ &\quad +\alpha_n\tan\alpha_n h)+4\alpha_m\alpha_n\tan\alpha_m h\tan\alpha_n h] \\ &= \tfrac{1}{4}\omega_0\phi_m(0)\phi_n(0)[6\omega_0^4+(\alpha_m+\alpha_n)^2+2\alpha_m\alpha_n], \end{aligned} \tag{A 4}$$

$$\begin{aligned} f_5(\phi_1,\phi_m) &= \tfrac{1}{2}\omega_0\phi_1(0)\phi_m(0)[\alpha_m^2+1-\omega_0^2(\tanh h+\alpha_m\tan\alpha_m h)] \\ &= \tfrac{1}{2}\omega_0\phi_1(0)\phi_m(0)[\alpha_m^2+1], \end{aligned} \tag{A 5}$$

correct to order ϵ.

The nonlinear, inhomogeneous wavemaker interaction terms in (25b) may be defined using (18c) by the following:

$$\begin{aligned} W_1\left(\phi_1,\xi,\frac{z}{h}\right) &= \frac{\partial}{\partial(z/h)}\left[\phi_1'\left(\frac{z}{h}\right)\left(\frac{S^*}{A^*}\right)\xi\left(\frac{z}{h}\right)\right] \\ &= \omega_0^{-1}\left\{a_1\left(\phi_1\left(\frac{z}{h}\right)\phi_1'\left(\frac{z}{h}\right)\right)' - \sum_{m=2} a_m\alpha_m\left(\phi_1'\left(\frac{z}{h}\right)\phi_m\left(\frac{z}{h}\right)\right)'\right\}, \end{aligned} \tag{A 6}$$

$$\begin{aligned} W_2\left(\phi_m,\xi,\frac{z}{h}\right) &= \frac{\partial}{\partial(z/h)}\left[\phi_m'\left(\frac{z}{h}\right)\left(\frac{S^*}{A^*}\right)\xi\left(\frac{z}{h}\right)\right] \\ &= \omega_0^{-1}\left\{a_1\left(\phi_1\left(\frac{z}{h}\right)\phi_m'\left(\frac{z}{h}\right)\right)' - \sum_{n=2} a_n\alpha_n\left(\phi_n\left(\frac{z}{h}\right)\phi_m'\left(\frac{z}{h}\right)\right)'\right\}. \end{aligned} \tag{A 7}$$

The inner product terms that are required to compute the coefficients of the eigenfunctions used in the second-order potentials ${}_2\Phi^{\mathrm{f}}(x,z,\tau)$ and ${}_2\Psi^{\mathrm{f}}(x,z)$ are defined by the integral

$$\langle\cdot\,,\cdot\rangle_{z/h} = \int_{-1}^{0}(\cdot\,,\cdot)\,\mathrm{d}\left(\frac{z}{h}\right),$$

which may be integrated to obtain

$$\begin{aligned} \alpha_n\langle(\phi_m\phi_n')',\Lambda_j\rangle_{z/h} &= \frac{\phi_m(0)\phi_n(0)\Lambda_j(0)}{\mathscr{D}(\alpha_m,\alpha_n,\lambda_j)}\{\omega_0^2[2\lambda_j^2\alpha_n^2+\lambda_j^2(\alpha_m^2+\alpha_n^2) \\ &\quad -(\alpha_m^2-\alpha_n^2)^2]+\Omega_j[\alpha_n^2(\alpha_n^2-\alpha_m^2-\lambda_j^2)+\omega_0^4(\alpha_n^2-\alpha_m^2+\lambda_j^2)]\}, \end{aligned} \tag{A 8}$$

where

$$\Omega_j = -\lambda_j\tan\lambda_j h = \begin{Bmatrix} 4\omega_0^2 & \text{for} & \Lambda_j(z/h) = Q_j(z/h) & \text{and} & \lambda_j = \beta_j \\ 0 & \text{for} & \Lambda_j(z/h) = \psi_j(z/h) & \text{and} & \lambda_j = \mu_j \end{Bmatrix},$$

and

$$\begin{aligned} \mathscr{D}(\alpha_m,\alpha_n,\lambda_j) &= [(\alpha_m+\alpha_n)^2-\lambda_j^2][(\alpha_m-\alpha_n)^2-\lambda_j^2] \\ &= [(\alpha_m^2+\alpha_n^2-\lambda_j^2)+2\alpha_m\alpha_n][(\alpha_n^2+\alpha_m^2-\lambda_j^2)-2\alpha_m\alpha_n]. \end{aligned}$$

Note that for $n = 1$, $\alpha_1 = \mathrm{i}$ and

$$\phi_1' = \frac{\sin[\mathrm{i}h(1+z/h)]}{n_1} = \frac{\mathrm{i}\sinh[h(1+z/h)]}{n_1},$$

so that
$$\alpha_n\langle(\phi_m\phi_n')', A_j\rangle_{z/h} = -\langle(\phi_m\phi_1')', A_j\rangle_{z/h}; \quad n = 1.$$

Similarly, the inner product between ϕ_1' and ϕ_m' required in order to compute the dimensionless mean horizontal momentum per unit area may be recovered from (A 8) as

$$\alpha_m\langle\phi_1'\phi_m'\rangle_{z/h} = \alpha_m\langle(\phi_1\phi_m')', \psi_0\rangle_{z/h} = -\omega_0^2\phi_1(0)\,\phi_m(0).$$

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